

I. INTRODUCTION

Special relativity is foundation of many branches of modern physics, of which theoretical results are far beyond our daily experience and hard to realized in kinematic experiments. Nevertheless, squeezed light may supply a possible experimental test in modern optics laboratories. By using canonical transformation of Wigner distribution function in phase space, the theoretical scheme of possible experimental tests for Wigner rotation and Thomas precession were put forward [1, 2]. However, the additional law of relativistic velocities, which is distinct from the usual additional law of vectors, had not been obtained. In present paper, we not only acquire the corresponding additional law but also use a more direct method, which will enclose the essence of the analogue.

As a nonclassical light field, squeezed light has a fundamental role in the development of quantum optics, which preserves the minimum-uncertainty product in the phase space and may exhibit many interesting properties such as sub-poissonian photon counting statistics and photon antibunching [3, 4]. Its nonlinear generalization and the corresponding features were studied by Kwek and Kiang [5]. In addition, its noncyclic and nonunitary geometric phase were formulated by Yang et. al. [6].

This article is organized as follows. Sec. II reviews some of indispensable concepts of squeezed optics as well as special relativity necessary for the present paper. In Sec. III, the connection between the squeezed operator and Lorentz boost is built. In Sec. IV, the additional law of relativistic velocities and the angle of Wigner rotation are deduced as well. Moreover, a possible experimental test on the additional law of relativistic velocities is put forward. At the end of this paper, a conclusion is drawn.

II. REVIEWS OF SQUEEZED STATE AND SPECIAL RELATIVITY

Squeezed state operator

$$S(\beta) = \exp\left(\frac{1}{2}\beta a^{\dagger 2} - \frac{1}{2}\beta^* a^2\right) \quad (1)$$

can be regarded as coset space of $SU(1,1)$, which is $SU(1,1)/U(1)$, where β is a complex number, $*$ denotes the conjugate operation and a^\dagger and a are creation and annihilation operators respectively that satisfy the canonical commutation relation $[a, a^\dagger] = 1$. In order to demonstrate this idea clearly, let us review the necessary knowledge about $SU(1,1)$ [7]. It's generators satisfy the following commutation relations, which are

$$[K_1, K_2] = -iK_0, \quad [K_0, K_1] = iK_2, \quad [K_0, K_2] = -iK_1. \quad (2)$$

Via choosing another appropriate basis, the generators become

$$K_\pm = \pm i(K_1 \pm iK_2), \quad K_0.$$

Therefore, the commutation relations are transformed to be

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \quad (3)$$

Furthermore, the definition of Perelomov's $SU(1,1)$ coherent state is given:

$$S(\beta) = \exp(\beta K_+ - \beta^* K_-). \quad (4)$$

Last but not least, the boson realization of $SU(1,1)$ is supplied, so the generators become

$$K_+ = \frac{1}{2}a^{\dagger 2}, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{4}(aa^\dagger + a^\dagger a). \quad (5)$$

which satisfy the commutation relations of (3). Substituting Eq. (5) into Eq. (4), the squeezed operator (1) is recovered. As $SU(1,1)$ is locally isomorphic to $SO(2,1)$, the coherent state of $SU(1,1)$ in another word the squeezed

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operator, may have connection with the Lorentz boost, which will be proved in Sec. III. Moreover, the following paragraph will review the indispensable knowledge about special relativity.

Let us consider about $(2+1)$ -dimensional spacetime. There are two inertial frames of reference Σ and Σ' that are coincident at time $t = 0$ and the velocity of Σ' relative to Σ is $\mathbf{v} = (v^1, v^2)$. An event is observed as $(x^0, x^1, x^2)^T$ in Σ and $(y^0, y^1, y^2)^T$ in Σ' . The two teams of coordinates is connected by a Lorentz boost \mathcal{L} , i.e. $(y^0, y^1, y^2)^T = \mathcal{L}(x^0, x^1, x^2)^T$, where $x^0 = ct$, T denotes the operation of transposition and \mathcal{L} [8] takes the form

$$\mathcal{L} = \begin{pmatrix} \gamma & -\eta\gamma\frac{v^1}{v} & -\eta\gamma\frac{v^2}{v} \\ -\eta\gamma\frac{v^1}{v} & 1 + \eta^2\frac{\gamma^2}{1+\gamma}(\frac{v^1}{v})^2 & \eta^2\frac{\gamma^2}{1+\gamma}\frac{v^1}{v}\frac{v^2}{v} \\ -\eta\gamma\frac{v^2}{v} & \eta^2\frac{\gamma^2}{1+\gamma}\frac{v^1}{v}\frac{v^2}{v} & 1 + \eta^2\frac{\gamma^2}{1+\gamma}(\frac{v^2}{v})^2 \end{pmatrix}, \quad (6)$$

where $v = \sqrt{(v^1)^2 + (v^2)^2}$, $\eta = v/c$ and $\gamma = 1/\sqrt{1-\eta^2}$. To simplify above Eq. (6), let us introduce the auxiliary variables which are $\cosh \rho = \gamma$, $\sinh \rho = \eta\gamma$, $\cos \varphi = v^1/v$ and $\sin \varphi = v^2/v$, so that Eq. (6) becomes

$$\begin{pmatrix} \cosh \rho & -\sinh \rho \cos \varphi & -\sinh \rho \sin \varphi \\ -\sinh \rho \cos \varphi & 1 + (\cosh \rho - 1) \cos^2 \varphi & (\cosh \rho - 1) \cos \varphi \sin \varphi \\ -\sinh \rho \sin \varphi & (\cosh \rho - 1) \cos \varphi \sin \varphi & 1 + (\cosh \rho - 1) \sin^2 \varphi \end{pmatrix}, \quad (7)$$

where the formulae $\cosh^2 \rho - \sinh^2 \rho = 1$ and $\sinh^2 \rho / (1 + \cosh \rho) = \cosh \rho - 1$ are useful. If $\varphi = 0$ or $\pi/2$, readers can check that the common Lorentz boost which always appears in the standard text book about special relativity is reduced.

Further, let us recall additional law of relativistic velocities. Σ , Σ' and Σ'' are arbitrary three inertial frames of reference that are coincident at the initial time. If the velocity of Σ' relative to Σ is \mathbf{u} and the velocity of Σ'' relative to Σ' is \mathbf{v} , the velocity of Σ'' relative to Σ is $\mathbf{u} \oplus \mathbf{v}$, where \oplus is the additional operation of relativistic velocities which is distinct from the parallelogram law of velocities. Its concrete formula [8] reads

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left[\mathbf{u} + \frac{1}{\gamma_u} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right]. \quad (8)$$

Observers at rest relative to Σ (relative to Σ') agree with observers at rest relative to Σ' (relative to Σ'') that their space time coordinate are linked by a pure Lorentz transformation without rotations. However, if the two successive Lorentz boost are in noncollinear directions, observers at rest relative to Σ agree with observers at rest relative to Σ'' that their coordinate are linked not only by a Lorentz transformation but also by a rotation, which is called Wigner rotation.

III. CONNECTING SQUEEZED TRANSFORMATION WITH LORENTZ BOOST

In this section, our main goal is to prove that the squeezed transformation corresponds to Lorentz boost. To begin with, we express a vector (x^0, x^1, x^2) in $(2+1)$ -dimensional space time as

$$x = x^0 K_0 - x^1 K_1 - x^2 K_2, \quad (9)$$

whose basis are (K_0, K_1, K_2) , which are the generators of $SU(1, 1)$. Moreover, let us do this operation,

$$y^0 K_0 - y^1 K_1 - y^2 K_2 = S(\beta)(x^0 K_0 - x^1 K_1 - x^2 K_2)S^\dagger(\beta). \quad (10)$$

In order to get specific connection between (y^0, y^1, y^2) and (x^0, x^1, x^2) , we must calculate the r.h.s. of Eq. (10). During the calculation, the BCH formula [9] may be useful, which takes the form

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \quad (11)$$

By using the above Eq. (11) and the commutations relations (2) of $SU(1, 1)$, one can get

$$\begin{aligned} S(\beta)K_0S^\dagger(\beta) &= \cosh \rho K_0 + \cos \varphi \sinh \rho K_1 + \sin \varphi \sinh \rho K_2 \\ S(\beta)K_1S^\dagger(\beta) &= \cos \varphi \sinh \rho K_0 + [1 + (\cosh \rho - 1) \cos^2 \varphi] K_1 \\ &\quad + (\cosh \rho - 1) \cos \varphi \sin \varphi K_2, \\ S(\beta)K_2S^\dagger(\beta) &= \sin \varphi \sinh \rho K_0 + (\cosh \rho - 1) \cos \varphi \sin \varphi K_1 \\ &\quad + [1 + (\cosh \rho - 1) \sin^2 \varphi] K_2 \end{aligned} \quad (12)$$

where

$$\beta = \frac{\rho}{2}e^{i(\pi/2-\varphi)} = \frac{\rho}{2}(\sin \varphi + i \cos \varphi). \quad (13)$$

Substituting above Eq. (12) into Eq. (10), one gains that

$$\begin{aligned} & y^0 K_0 - y^1 K_1 - y^2 K_2 \\ = & [\cosh \rho x^0 - \sinh \rho \cos \varphi x^1 - \sinh \rho \sin \varphi x^2] K^0 \\ & - \{-\sinh \rho \cos \varphi x^0 + [1 + (\cosh \rho - 1) \cos^2 \varphi] x^1 + [(\cosh \rho - 1) \cos \varphi \sin \varphi] x^2\} K^1 \cdot \\ & - \{-\sinh \rho \sin \varphi x^0 + [(\cosh \rho - 1) \cos \varphi \sin \varphi] x^1 + [1 + (\cosh \rho - 1) \sin^2 \varphi] x^2\} K^2 \end{aligned}$$

Because the linear independence of the generators, the above equation can be written as a matrix form, which is

$$\begin{pmatrix} y^0 \\ y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} \cosh \rho & -\sinh \rho \cos \varphi & -\sinh \rho \sin \varphi \\ -\sinh \rho \cos \varphi & 1 + (\cosh \rho - 1) \cos^2 \varphi & (\cosh \rho - 1) \cos \varphi \sin \varphi \\ -\sinh \rho \sin \varphi & (\cosh \rho - 1) \cos \varphi \sin \varphi & 1 + (\cosh \rho - 1) \sin^2 \varphi \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix}. \quad (14)$$

By comparison with Eq. (7), it is found that the above (3×3) coefficient matrix is identical with Lorentz boost (7). So it has been proved that $S(\beta)xS^\dagger(\beta)$ is can be regarded as Lorentz boost under the condition that $\beta = \frac{\rho}{2}(\sin \varphi + i \cos \varphi)$. If we do the above calculations on the boson representation, the Lorentz boost can realized by the above operation of quantum optics.

IV. SQUEEZED TRANSFORMATION AND ADDITIONAL LAW OF RELATIVISTIC VELOCITIES

Furthermore, the next task is to demonstrate the additional law of relativistic velocities by use of squeezed operator in this section. We start out from this formula

$$S(\beta_1)S(\beta_2) = S(\beta_3)R(\delta), \quad (15)$$

where $R(\delta) = \exp(iK_0\delta) = \exp[i\frac{1}{4}(2a^\dagger a + 1)\delta]$. From the two previous sections, we get the idea that Eq. (15) means that two Lorentz boosts is equal to not only a boost but also a rotation. In order to represent β_3 and δ by use of β_1 and β_2 , let's do the following operations on Eq. (15), i.e. ,

$$S(\beta_1)S(\beta_2)aS(\beta_2)^\dagger S(\beta_1)^\dagger = S(\beta_3)R(\delta)aR(\delta)^\dagger S(\beta_3)^\dagger. \quad (16)$$

With the help of the equations

$$\begin{aligned} S(\beta)aS(\beta)^\dagger &= a \cosh \frac{\rho}{2} - a^\dagger e^{i(\frac{\pi}{2}-\varphi)} \sinh \frac{\rho}{2} \\ S(\beta)a^\dagger S(\beta)^\dagger &= a^\dagger \cosh r - a e^{-i\theta} \sinh r, \\ R(\delta)aR(\delta)^\dagger &= a e^{-i\frac{\delta}{2}} \end{aligned}$$

l.h.s. and r.h.s. of Eq. (16) can be written as linear combination of a and a^\dagger respectively. As a and a^\dagger are linear independent, the coefficient of a of l.h.s. of Eq. (16) coincides with counterpart of r.h.s., and the same is true for a^\dagger . Hence, we obtain

$$a : \quad \cosh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + e^{i(\varphi_1-\varphi_2)} \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2} = e^{-i\frac{\delta}{2}} \cosh \frac{\rho_3}{2}, \quad (17)$$

and

$$a^\dagger : \quad e^{-i\varphi_1} \sinh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + e^{-i\varphi_2} \sinh \frac{\rho_2}{2} \cosh \frac{\rho_1}{2} = e^{-i\frac{\delta}{2}} e^{-i\varphi_3} \sinh \frac{\rho_3}{2}. \quad (18)$$

Moreover, the above two complex Eq. (17) and Eq. (18) can be converted into four real equations, which are

$$\cosh \frac{\rho_3}{2} \cos \frac{\delta}{2} = \cosh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \cos(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2}, \quad (19)$$

$$\sin \frac{\delta}{2} \cosh \frac{\rho_3}{2} = \sinh(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2}, \quad (20)$$

$$\cos(\theta_3 + \frac{\delta}{2}) \sinh \frac{\rho_3}{2} = \cos \varphi_1 \sinh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \cos \varphi_2 \sinh \frac{\rho_2}{2} \cosh \frac{\rho_1}{2} \quad (21)$$

and

$$\sin(\theta_3 + \frac{\delta}{2}) \sinh \frac{\rho_3}{2} = \sin \varphi_1 \sinh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \sin \varphi_2 \sinh \frac{\rho_2}{2} \cosh \frac{\rho_1}{2}. \quad (22)$$

By use of Eq. (19) and Eq. (20), one can reach the following results

$$\tan \frac{\delta}{2} = \frac{\sinh(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2}}{\cosh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \cos(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2}} \quad (23)$$

and

$$\cosh^2 \frac{\rho_3}{2} = \sinh^2 \frac{\rho_1}{2} \sinh^2 \frac{\rho_2}{2} + \cosh^2 \frac{\rho_1}{2} \cosh^2 \frac{\rho_2}{2} + 2 \cos(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2} \cosh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2}. \quad (24)$$

We should note that Eq. (23) is just the formula for Wigner angle [10]. Substituting the following formulae $\cosh 2x = \cosh^2 x + \sinh^2 x$, $\cosh^2 x - \sinh^2 x = 1$ and $\sinh 2x = 2 \sinh x \cosh x$ into Eq.(24), we can get a more elegant and pragmatic expression about $\cosh \rho_3$, which is

$$\cosh \rho_3 = \cosh \rho_1 \cosh \rho_2 + \cos(\varphi_2 - \varphi_1) \sinh \rho_1 \sinh \rho_2. \quad (25)$$

In addition, let us introduce the following auxiliary variables

$$\begin{aligned} \gamma_u &= \cosh \rho_1 & \gamma_v &= \cosh \rho_2 & \gamma_w &= \cosh \rho_3 \\ \mathbf{u} &= c \tanh \rho_1 \hat{u} & \mathbf{v} &= c \tanh \rho_2 \hat{v} & \mathbf{w} &= c \tanh \rho_3 \hat{w} \end{aligned} \quad (26)$$

where \hat{u} , \hat{v} and \hat{w} are unit vectors along the direction of \mathbf{u} , \mathbf{v} and \mathbf{w} respectively and c is the velocity of light propagating in the vacuum. Substituting Eq. (26) into Eq. (25), we can get a more meaningful formula, which is

$$\gamma_w = \gamma_u \gamma_v (1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}) \quad (27)$$

Now, we concentrate on another two equations (21) and (22), by substituting $\cos \delta/2$ from Eq. (19) and $\sin \delta/2$ from Eq. (20) into them, then they compose a set of equations about unknown elements $\cos \varphi_3$ and $\sin \varphi_3$, i.e. ,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \cos \varphi_3 \\ \sin \varphi_3 \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix}, \quad (28)$$

where $A = -D = \tanh \frac{\rho_3}{2} \sin(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2}$, $B = C = \tanh \frac{\rho_3}{2} [\cosh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \cos(\varphi_2 - \varphi_1) \sinh \frac{\rho_1}{2} \sinh \frac{\rho_2}{2}]$, $E = \cos \varphi_1 \sinh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \cos \varphi_2 \sinh \frac{\rho_2}{2} \cosh \frac{\rho_1}{2}$ and $F = \sin \varphi_1 \sinh \frac{\rho_1}{2} \cosh \frac{\rho_2}{2} + \sin \varphi_2 \sinh \frac{\rho_2}{2} \cosh \frac{\rho_1}{2}$. According to Cramer's rules, the above Eq. (28) can be solved, hence we can obtain

$$\cos \varphi_3 = \frac{\begin{vmatrix} E & B \\ F & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{\frac{1}{\sinh \rho_3} \{ [\sinh \rho_1 \cosh \rho_2 + \sinh \rho_2 (\cosh \rho_1 - 1) \times (\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2)] \cos \varphi_1 + \sinh \rho_2 \cos \varphi_2 \}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} \quad (29)$$

and

$$\sin \varphi_3 = \frac{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{\frac{1}{\sinh \rho_3} \{ [\sinh \rho_1 \cosh \rho_2 + \sinh \rho_2 (\cosh \rho_1 - 1) \times (\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2)] \sin \varphi_1 + \sinh \rho_2 \sin \varphi_2 \}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}. \quad (30)$$

In order to disclose the meaning of additional law of relativistic velocities. Let us introduce three unit vectors, which are

$$\hat{u} = \cos \varphi_1 \hat{i} + \sin \varphi_1 \hat{j} \quad \hat{v} = \cos \varphi_2 \hat{i} + \sin \varphi_2 \hat{j} \quad \text{and} \quad \hat{w} = \cos \varphi_3 \hat{i} + \sin \varphi_3 \hat{j}.$$

Hence Eq. (29) and Eq. (30) become

$$\hat{w} = \frac{1}{\sinh \rho_3} \{ [\sinh \rho_1 \cosh \rho_2 + \sinh \rho_2 (\cosh \rho_1 - 1) \hat{u} \cdot \hat{v}] \hat{u} + \sinh \rho_2 \hat{v} \}. \quad (31)$$

Moreover, substituting Eq. (26) and Eq. (27) into Eq. (31), the final addition law

$$\mathbf{w} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left(\mathbf{u} + \frac{\gamma_u}{\gamma_u + 1} \frac{\mathbf{u} \cdot \mathbf{v}}{c^2} \mathbf{u} + \frac{1}{\gamma_u} \mathbf{v} \right) \quad (32)$$

is achieved, which is identical with Eq. (8). When $c \rightarrow \infty$, the common additional law of velocities

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

in Galilean transformation is obtained.

Depending on the analysis above, the additional law of relativistic velocities has been successfully achieved from squeezed optics. Furthermore, the theoretical results can be possibly verified by the following experiment. In practice, it is much easier to realize the two-mode squeezed state

$$S^{\text{tm}}(\beta) = \exp\left[\frac{1}{2}(\beta a_1^\dagger a_2^\dagger + \beta^* a_1 a_2)\right]$$

than one-mode one, where subscripts 1 and 2 denote the two different photons and superscript “tm” represents two-mode, thus we choose two-mode process. Because the generators of two-mode squeezed operator obey the same Lie algebra as the one-mode squeezed operator, our analysis above also applies. At first, a beam of light was splitted into two beams. Second, beam 1 goes through a nondegenerate optical parametric amplifier (NOPA) which could generate two-mode squeezed vacuum state [11]. As a result, beam 1 becomes $S^{\text{tm}}(\beta_2)|00\rangle$. Moreover, it continues to pass an equipment which makes the light become $S^{\text{tm}}(\beta_1)S^{\text{tm}}(\beta_2)|00\rangle$. Up to beam 2, it is changed to be $S^{\text{tm}}(\beta_3)|00\rangle$ by NOPA, where β_3 is calculated from β_1 and β_2 according to Eq. (25), (29) and (30). After all the transformations on beams 1 and 2, let them interfere with each other. Then the dark and bright fringes can be observed. The separation of adjacent bright fringes gives δ' . If δ' coincides with δ of Eq. (23), then the additional law of relativistic velocities is demonstrated.

V. CONCLUSION AND ACKNOWLEDGMENTS

In summary, since squeezed transformations constitute coset space of $SU(1,1)$, Lorentz transformations make up Lorentz group and $SU(1,1)$ group is locally isomorphic to the $(2+1)$ -dimensional Lorentz group [7], we demonstrate the phenomenon of special relativity by use of squeezed optics. At first it is proved that the squeezed transformation (10) is equivalent to Lorentz boost (14) under the condition (13). Furthermore, the additional law of relativistic velocities (32) and the angle of Wigner rotation (23) are deduced as well. Specifically speaking, the relations between squeezed parameters ($\beta_1 = \frac{\rho_1}{2}e^{i(\pi/2-\varphi_1)}$, $\beta_2 = \frac{\rho_2}{2}e^{i(\pi/2-\varphi_2)}$ and $\beta_3 = \frac{\rho_3}{2}e^{i(\pi/2-\varphi_3)}$ in Eq. (15)) and velocities (\mathbf{u} , \mathbf{v} and \mathbf{w} in Eq. (32)) are illustrated below,

velocity	modulo	directional vector
\mathbf{u}	$c \tanh \rho_1$	$(\cos \varphi_1, \sin \varphi_1)$
\mathbf{v}	$c \tanh \rho_2$	$(\cos \varphi_2, \sin \varphi_2)$
\mathbf{w}	$c \tanh \rho_3$	$(\cos \varphi_3, \sin \varphi_3)$

Moreover, a possible experimental test on the additional law of relativistic velocities is also discussed.

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